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SOME BASIC ASPECTS OF OCEANIC INTERNAL WAVES AND THE SOUND-SPEE--ETC(U)

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# Some Basic Aspects of Oceanic Internal Waves and the Sound-Speed Fluctuations They Generate

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## PREFACE

This short summary of oceanic internal waves and their effect on the speed of sound was prepared as an appendix to an interim report concerned with the influence of internal waves on the performance of acoustic surveillance systems. Because of the limited distribution of the report it seems worthwhile to make the material available separately. Hopefully, it will assist the reader in understanding the many papers which have recently appeared dealing with the subject of acoustic fluctuations and ocean variability. It is concerned with deriving as rapidly and as directly as possible the mathematical expression for the correlation function of the sound-speed fluctuations resulting from internal waves. As a regrettable consequence, a great deal of the theory of internal waves is left by the wayside.

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## SOME BASIC ASPECTS OF OCEANIC INTERNAL WAVES AND THE SOUND-SPEED FLUCTUATIONS THEY GENERATE

Internal waves are those oscillations in the ocean which result from a stable density (entropy) stratification. They are thought to be responsible for motions with horizontal scales ranging from kilometers to tens of kilometers, vertical scales ranging from tens of meters to hundreds of meters, and temporal scales ranging from minutes to hours. Experimental studies<sup>1</sup> conducted over a period of years indicate internal waves exist in all the world's oceans, have a characteristic (universal) spectrum which is stationary, horizontally homogeneous, and, to a large degree, horizontally isotropic.

Two developments have recently occurred which greatly enhance our ability to discuss, in quantitative terms, the relationship between internal waves and acoustic fluctuations. These important advances are (1) the construction in 1972 of a general, phenomenological internal-wave model by Garrett and Munk<sup>2</sup> and (2) the establishment in 1975 by Munk and Zachariasen<sup>3</sup> of the precise relationship between the quantities which characterize the internal-wave field and the induced change in the sound-speed profile. In this report these two developments will be reviewed. To this aim we only consider those aspects of the Garrett-Munk (GM) model which are relevant for acoustic signal coherence studies.

We begin with the linear equations which govern internal waves, i.e.,

$$\frac{\partial u_x}{\partial t} + \omega_i u_y + \frac{1}{\rho_0} \frac{\partial p_{iw}}{\partial x} = 0 \quad (1a)$$

$$\frac{\partial u_y}{\partial t} - \omega_i u_x + \frac{1}{\rho_0} \frac{\partial p_{iw}}{\partial y} = 0, \quad (1b)$$

D. R. PALMER

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0} \frac{\partial p_{IW}}{\partial z} + n^2(z) \zeta = 0, \quad (1c)$$

$$\frac{\partial \zeta}{\partial t} = v, \quad (1d)$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial v}{\partial z} = 0, \quad (1e)$$

where

$u_x, u_y$  = horizontal components of particle velocity,

$v$  = vertical component of particle velocity,

$p_{IW}$  = internal-wave induced pressure,

$\zeta$  = vertical particle displacement,

$\rho_0$  = equilibrium density (taken to be constant),

$n(z)$  = bouyancy or Brunt-Väisälä (B-V) frequency,

$\omega_i$  = inertial frequency,

= (2 cycles/day)sin(latitude),

= 0.042 cph at 30° latitude.

The coordinate system is orientated so that the  $z$  axis is positive downward, the  $x$  axis (the direction of  $u_x$ ) is to the east and the  $y$  axis (the direction of  $u_y$ ) is to the south.

These equations are the result of several approximations. One starts with the source-free hydrodynamic equations valid for a rotating earth. The equations are then linearized and the Boussinesq approximation is made. It is then assumed that the cosine of the latitude may be set equal to zero. (This last approximation is called the "traditional" approximation.) The details of the derivation of Eqs. (1) may be culled from the book by Phillips<sup>4</sup> or the one by Eckart.<sup>5</sup>



Equations (1) are linear, coupled differential equations and can therefore be solved by Fourier transforming in time and then using the technique of separation of variables. The general solution is

$$\begin{bmatrix} u_x(\mathbf{x}, t) \\ u_y(\mathbf{x}, t) \\ v(\mathbf{x}, t) \\ \zeta(\mathbf{x}, t) \\ p_{IW}(\mathbf{x}, t)/\rho_0 \end{bmatrix} = \sum_j \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int_0^{2\pi} \frac{d\psi}{(2\pi)} A_j(\psi, \omega) \exp[ik_j r \cos(\psi - \phi) - i\omega t] \begin{bmatrix} U_x \\ U_y \\ V \\ D \\ P \end{bmatrix}, \quad (2a)$$

where

$$\begin{bmatrix} U_x \\ U_y \\ V \\ D \\ P \end{bmatrix} = \begin{bmatrix} \frac{1}{k_j} Z_j'(z) (i \cos \psi + \frac{\omega_i}{\omega} \sin \psi) \\ \frac{1}{k_j} Z_j'(z) (i \sin \psi - \frac{\omega_i}{\omega} \cos \psi) \\ Z_j(z) \\ \frac{i}{\omega} Z_j(z) \\ \frac{i(\omega^2 - \omega_i^2)}{\omega k_j^2} Z_j'(z) \end{bmatrix}. \quad (2b)$$

Here  $x = r \cos \phi$ ,  $y = r \sin \phi$  and the  $A_j$ 's are constant coefficients (i.e., they are independent of  $x$  and  $t$ ). The normal-mode eigenfunctions  $Z_j(z)$  and corresponding eigenvalues  $k_j$  are determined by the eigenvalue equation

$$Z_j''(z) + k_j^2 \left( \frac{n^2(z) - \omega^2}{\omega^2 - \omega_i^2} \right) Z_j(z) = 0. \quad (3)$$

Here, and in Eq. (2b), a prime indicates differentiation with respect to  $z$ . The eigenfunctions are assumed to vanish at the surface (i.e., the top of the main thermocline) and at the bottom of the ocean. The index  $j$  is discrete and is taken to run from one to infinity.

For sound coherence studies the effect of the vertical displacement is more important than that of the pressure or the components of the velocity.<sup>3,6</sup> Consequently, we need only consider the expression for  $\zeta$

$$\zeta(\mathbf{x}, t) = \sum_j \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int_0^{2\pi} \frac{d\psi}{(2\pi)} \left[ \frac{i}{\omega} \right] A_j(\psi, \omega) \exp[ik_j r \cos(\psi - \phi) - i\omega t] Z_j(z). \quad (4)$$

In the GM model, the amplitudes  $A_j(\psi, \omega)$  are taken to be zero-mean, Gaussian random variables. Hence, all moments of the  $A_j$ 's can be expressed in terms of the correlation function  $\langle A_j(\psi, \omega) A_{j'}^*(\psi', \omega') \rangle$ . (It is possible to use the reality of  $\zeta$  to relate  $\langle A_j A_{j'} \rangle$  to  $\langle A_j A_{j'}^* \rangle$ .) The requirements of reality, stationarity, and horizontal homogeneity dictate that this correlation function have the general form

$$\langle A_j(\psi, \omega) A_{j'}^*(\psi', \omega') \rangle = (2\pi)^2 \delta(\omega - \omega') \delta(\psi - \psi') \delta_{jj'} \Gamma_j(\psi, \omega) \quad (5)$$

where  $\Gamma_j$  is a real function having the property

$$\Gamma_j(\psi, \omega) = \Gamma_j(\psi \pm \pi, -\omega).$$

If the additional assumption of horizontal isotropy is made,  $\Gamma_j(\psi, \omega)$  is independent of the angle  $\psi$

$$\Gamma_j(\psi, \omega) = \Gamma_j(\omega) = \Gamma_j(-\omega). \quad (6)$$

With Eqs. (4)-(6) we have

$$\langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t') \rangle = \frac{1}{\pi} \sum_j \int_0^{\infty} \frac{d\omega}{\omega^2} \Gamma_j(\omega) J_0(k_j R) \cos[\omega(t - t')] Z_j(z) Z_j(z'). \quad (7)$$

Here  $R = \sqrt{(x - x')^2 + (y - y')^2}$  and  $J_0$  is the common Bessel function. We have used the fact that  $\Gamma_j$ ,  $Z_j$ , and  $k_j$  are all even functions of  $\omega$  to write the integral as one over positive frequencies only. Since the  $A_j$ 's are Gaussian random variables, so is  $\zeta$ , and all moments of  $\zeta$  can be expressed in terms of the correlation function given in Eq. (7).

Garrett and Munk assumed a "canonical form" for the buoyancy frequency,

$$n(z) = n_0 e^{-z/B}, \quad (8)$$

which is thought to give a reasonable representation below the mixed layer. Typically,  $n_0 = 3$  cph and  $B = 1.3$  km. They then solved for the modal eigenfunctions and eigenvalues using the WKB approximation with the result that



$$Z_j(z) \approx \left( \frac{2C_j}{n(z)} \right)^{1/2} \sin \left( \frac{j\pi n(z)}{n_0} \right); \omega_i < \omega < n(z),$$

$$\approx 0; \text{ otherwise,} \quad (9)$$

where the  $C_j$ 's are normalization constants and

$$k_j = \frac{\pi j}{Bn_0} (\omega^2 - \omega_i^2)^{1/2}. \quad (10)$$

By assuming that  $|z - z'|/B$  is small and that short-period oscillations associated with variations in  $\bar{z} = \frac{1}{2}(z + z')$  are unobservable and can, therefore, be averaged out, one has

$$Z_j(z) Z_j(z') \approx \frac{C_j}{n(\bar{z})} \cos [\kappa_j(\bar{z})(z - z')], \quad (11)$$

for  $\omega_i < \omega < n(\bar{z})$  and zero otherwise. The quantity  $\kappa_j$  is called the local vertical wave number and is given by the expression

$$\kappa_j(z) = \frac{\pi j}{Bn_0} n(z). \quad (12)$$

Substituting Eq. (11) into Eq. (7) gives

$$\langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t') \rangle = - \frac{1/\pi}{n(\bar{z})} \sum_j \int_{\omega_i}^{n(\bar{z})} \frac{d\omega}{\omega^2} C_j \Gamma_j(\omega) J_0(k_j R) \cos [\omega(t - t')] \cos [\kappa_j(\bar{z})(z - z')]. \quad (13)$$

The mean-square vertical displacement is obtained from Eq. (13) by setting  $\mathbf{x} = \mathbf{x}'$  and  $t = t'$ , i.e.,

$$\langle (\zeta(\mathbf{x}, t))^2 \rangle = \frac{1/\pi}{n(z)} \sum_j \int_{\omega_i}^{n(z)} \frac{d\omega}{\omega^2} C_j \Gamma_j(\omega). \quad (14)$$

Because the internal-wave field is stationary and horizontally homogeneous  $\langle (\zeta(\mathbf{x}, t))^2 \rangle$  does not depend on  $t$  or on the horizontal coordinates  $x$  and  $y$ . It does depend on  $z$ , however, reflecting the vertical inhomogeneity of the internal-wave field. By setting

$$\langle (\zeta(\mathbf{x}, t))^2 \rangle \equiv \langle \zeta^2(z) \rangle$$

we can simplify the notation somewhat. It turns out that  $C_j \Gamma_j(\omega)/\omega^2$  rapidly decreases with increasing  $\omega$  so it is usually a valid approximation to set the upper limit of the integral in Eq. (14) equal to infinity. Hence

$$\langle \zeta^2(z) \rangle = \frac{(\text{constant})}{n(z)}, \quad (15)$$

i.e., the product  $n(z) \langle \zeta^2(z) \rangle$  is approximately independent of depth. This equation can be used to define an extrapolated mean-square displacement  $\langle \zeta_0^2 \rangle$ ,

$$n(z) \langle \zeta^2(z) \rangle = n_0 \langle \zeta_0^2 \rangle. \quad (16)$$

By fitting data Garrett and Munk found

$$\langle \zeta_0^2 \rangle^{1/2} \approx 7.3 \text{ m}. \quad (17)$$

Returning to Eq. (13), if we define a normalized spectrum by the relation

$$F_j(\omega, n(z)) = \frac{(C_j/\pi n(z)) \omega^{-2} \Gamma_j(\omega)}{\langle \zeta^2(z) \rangle}; \quad \omega_i < \omega < n(z), \\ = 0; \text{ otherwise} \quad (18)$$

we have, with  $\bar{z} = \frac{1}{2}(z + z')$ ,

$$\langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t') \rangle = \\ \langle \zeta^2(\bar{z}) \rangle \sum_j \int_0^\infty d\omega F_j(\omega, n(\bar{z})) J_0(k_j R) \cos[\kappa_j(\bar{z})(z - z')] \cos[\omega(t - t')], \quad (19)$$

and, clearly,

$$\sum_j \int_0^\infty d\omega F_j(\omega, n(z)) = 1. \quad (20)$$

The vertical displacement spectrum  $F_\zeta(\omega, j; z)$  defined by Munk and Zachariasen<sup>3</sup> is related to our  $F_j(\omega, n(z))$  through

$$F_\zeta(\omega, j; z) = \langle \zeta^2(z) \rangle F_j(\omega, n(z)). \quad (21)$$

Consequently

$$\langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t') \rangle = \\ \sum_j \int_{\omega_i}^{n(z)} d\omega F_\zeta(\omega, j; \bar{z}) J_0(k_j R) \cos[\kappa_j(\bar{z})(z - z')] \cos[\omega(t - t')], \quad (22)$$

and

$$\langle \zeta^2(z) \rangle = \sum_j \int_{\omega_j}^{n(z)} d\omega F_\zeta(\omega, j; z). \quad (23)$$

The integration in Eqs. (22) and (23) extends from  $\omega_j$  to  $n(z)$  as a result of the definition Eq. (18).

In GM-type models,  $F_j$  is the product of two factors

$$F_j(\omega, n) = H(j) G(\omega, n). \quad (24)$$

(We have explicitly indicated the dependence of  $G$  on  $n$  in writing Eq. (24). In most papers this dependence is suppressed.) It follows from the definition of  $F_j$  that

$$\sum_{j=1}^{\infty} H(j) = 1, \quad (25)$$

and

$$\int_0^{\infty} d\omega G(\omega, n) = 1. \quad (26)$$

The function  $G$  is taken to have the form

$$G(\omega, n) = N_G \frac{\omega_i (\omega^2 - \omega_i^2)^{1/2}}{\omega^3}; \quad \omega_i < \omega < n, \\ = 0; \text{ otherwise.} \quad (27)$$

The normalization constant  $N_G$  is determined from Eq. (26),

$$N_G = \frac{4}{\pi} \left[ 1 + O\left(\frac{\omega_i}{n}\right) \right] \approx \frac{4}{\pi}. \quad (28)$$

There are (at least) three versions of the GM model, GM72,<sup>2</sup> GM75,<sup>7</sup> and GM75 $\frac{1}{2}$ .<sup>8</sup> They differ in that they assume different forms for  $H(j)$ . The latest version, GM75 $\frac{1}{2}$ , assumes

$$H(j) = N_H \frac{1}{j^2 + j_*^2}, \quad (29)$$



where<sup>9</sup>

$$N_H^{-1} = \sum_{j=1}^{\infty} \frac{1}{j^2 + j_*^2} \approx \frac{1}{2j_*^2} (\pi j_* - 1). \quad (30)$$

Typically,  $j_* = 3$ .

Summarizing

$$\begin{aligned} \langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t') \rangle &= \left( \frac{n_0}{n(\bar{z})} \right) \langle \zeta_0^2 \rangle \\ &\times \sum_{j=1}^{\infty} \int_0^{\infty} d\omega H(j) G(\omega, n(\bar{z})) J_0(k_j R) \cos[\kappa_j(\bar{z})(z-z')] \cos[\omega(t-t')], \end{aligned} \quad (31)$$

where

$$\bar{z} = \frac{1}{2} (z+z'), \quad R = \sqrt{(x-x')^2 + (y-y')^2}, \quad (32a)$$

$$n(z) = n_0 e^{-z/B}, \quad (32b)$$

$$k_j = \frac{\pi j}{n_0 B} (\omega^2 - \omega_i^2)^{1/2}, \quad (32c)$$

$$\kappa_j(z) = \frac{\pi j}{n_0 B} n(z), \quad (32d)$$

$$\begin{aligned} G(\omega, n) &= \frac{4}{\pi} \omega_i \frac{(\omega^2 - \omega_i^2)^{1/2}}{\omega^3}; \quad \omega_i < \omega < n, \\ &= 0; \text{ otherwise,} \end{aligned} \quad (32e)$$

and

$$H(j) = \frac{(j^2 + j_*^2)^{-1}}{\sum_{j=1}^{\infty} (j^2 + j_*^2)^{-1}}. \quad (32f)$$

Typical values are

$$\omega_i = 0.042 \text{ cph},$$

$$n_0 = 3 \text{ cph},$$

$$B = 1.3 \text{ km},$$

$$\langle \zeta_0^2 \rangle^{1/2} = 7.3 \text{ m},$$

$$j_* = 3.$$

Before proceeding we would like to mention a caveat. Much of the theory of internal waves outlined here is dependent on the assumption that the buoyancy frequency is a mono-

tone decreasing function of depth below the mixed layer. There are situations, however, where the bouyancy frequency has a pronounced minimum which tends to trap the internal waves in a duct. This ducted propagation can have a pronounced effect on the character of the acoustic signal fluctuations.<sup>10</sup> For such a situation Eq. (9) is not correct and one might need to actually calculate the internal-wave normal-mode eigenfunctions numerically<sup>11</sup> in order to correctly describe the sound-speed fluctuations.

We now wish to record the relationship between the vertical particle displacement  $\zeta$  and the change in the sound speed. In the absence of internal-wave activity the sound speed is assumed to be a non-random function of depth alone

$$c(\mathbf{x}, t) = \langle c(\mathbf{x}, t) \rangle = c_0(z). \quad (33)$$

When internal waves are present the sound speed acquires a small fluctuation  $\delta c$  which is a random function of all three spatial coordinates and of the time

$$c(\mathbf{x}, t) = c_0(z) + \delta c(\mathbf{x}, t). \quad (34)$$

By definition  $\delta c$  has a mean value of zero

$$\langle \delta c(\mathbf{x}, t) \rangle = 0.$$

Munk and Zachariasen<sup>3</sup> argued that  $\delta c$  is the result of the vertical convection by the internal waves of the potential sound-speed gradient, i.e., the actual sound-speed gradient minus the adiabatic gradient

$$\delta c(\mathbf{x}, t) = [\partial_z c_0(z)]_p \zeta(\mathbf{x}, t), \quad (35)$$

where

$$[\partial_z c_0(z)]_p \equiv \partial_z c_0(z) - [\partial_z c_0(z)]_{\text{adiabatic}}. \quad (36)$$

The potential sound-speed gradient is convected rather than the measured gradient because internal waves are oscillations about the adiabatic background or reference state, i.e., internal waves exist because of entropy stratification. With some phenomenology Munk<sup>12</sup> and Munk and Zachariasen<sup>3</sup> showed that

$$[\partial_z c_0(z)]_p = -\frac{\mu}{g} c_0(z) n^2(z), \quad (37)$$

where  $g$  is the acceleration due to gravity,  $n(z)$  is, of course, the bouyancy frequency and  $\mu$  is a dimensionless number that can be expressed in terms of certain numbers which describe fractional changes in several physical quantities with changes in depth. For typical values of these parameters they found  $\mu = 24.5$ . Since  $\mu$  is subject to some variation this value should only be considered as representation. Desaubies,<sup>13</sup> for example, found  $\mu = 11.6$  for the sea water in the vicinity of Cobb Seamount. The fractional change in the sound speed due to the presence of internal waves follows from Eqs. (35) and (37)

$$\frac{\delta c}{c_0}(\mathbf{x}, t) = -\frac{\mu}{g} n^2(z) \zeta(\mathbf{x}, t). \quad (38)$$

The mean-square fractional sound-speed fluctuation can be obtained from Eqs. (16) and (38)

$$\left\langle \left( \frac{\delta c}{c_0}(\mathbf{x}, t) \right)^2 \right\rangle = \frac{\mu^2}{g^2} n_0 n^3(z) \langle \zeta_0^2 \rangle. \quad (39)$$

Just as with  $\zeta$  we set

$$\left\langle \left( \frac{\delta c}{c_0}(\mathbf{x}, t) \right)^2 \right\rangle \equiv \left\langle \left( \frac{\delta c}{c_0}(z) \right)^2 \right\rangle, \quad (40)$$

and define an extrapolated mean-square fluctuation  $\langle (\delta c/c)_0^2 \rangle$

$$\left\langle \left( \frac{\delta c}{c_0}(z) \right)^2 \right\rangle = \left( \frac{n(z)}{n_0} \right)^3 \left\langle \left( \frac{\delta c}{c} \right)_0^2 \right\rangle, \quad (41)$$

where

$$\left\langle \left( \frac{\delta c}{c} \right)_0^2 \right\rangle^{1/2} = \frac{n_0^2 \mu}{g} \langle \zeta_0^2 \rangle^{1/2}. \quad (42)$$

For the values  $n_0 = 3$  cph,  $\mu = 24.5$ , and  $\langle \zeta_0^2 \rangle^{1/2} = 7.3$  m we obtain

$$\left\langle \left( \frac{\delta c}{c} \right)_0^2 \right\rangle^{1/2} = 5 \times 10^{-4}. \quad (43)$$



Again,  $\zeta$  is a zero-mean, Gaussian random variable. Since the fractional sound-speed fluctuation  $(\delta c/c_0)(\mathbf{x}, t)$  is equal to a non-random, slowly varying function of depth times  $\zeta$ , it also is a zero-mean, Gaussian random variable. Hence its complete statistical description follows from the correlation function

$$\left\langle \frac{\delta c}{c_0}(\mathbf{x}, t) \frac{\delta c}{c_0}(\mathbf{x}', t') \right\rangle.$$

This function can be directly obtained from Eqs. (31), (38), (41) and (42)

$$\begin{aligned} \left\langle \frac{\delta c}{c_0}(\mathbf{x}, t) \frac{\delta c}{c_0}(\mathbf{x}', t') \right\rangle &= \left[ \frac{n(\bar{z})}{n_0} \right]^3 \left\langle \left[ \frac{\delta c}{c} \right]_0^2 \right\rangle \\ &\times \sum_{j=1}^{\infty} \int_0^{\infty} d\omega H(j) G(\omega, n(\bar{z})) J_0(k_j R) \cos[\kappa_j(\bar{z})(z - z')] \cos[\omega(t - t')]. \end{aligned} \quad (44)$$

Our goal here has been the derivation of Eq. (44). With it one has all that is needed of the GM model in order to describe the effects of internal waves on the sound-speed profile.

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11. See, for example, L.B. Dozier, "Statistics of Normal Mode Amplitudes in a Random Ocean," Ph. D. Thesis, New York University (1977); L.B. Dozier and F.D. Tappert, "Statistics of normal mode amplitudes in a random ocean. I. Theory," J. Acoust. Soc. Am. **63**, 353-365 (1978); L.B. Dozier and F.D. Tappert, "Statistics of normal mode amplitudes in a random ocean. II. Computations," J. Acoust. Soc. Am. **64**, 533-547 (1978).

NRL MEMORANDUM REPORT 4098

12. W.H. Munk, "Sound channel in an exponentially stratified ocean, with application to SOFAR," J. Acoust. Soc. Am. 55, 220-226 (1974).
13. Y.J.F. Desaubies, "Acoustic-phase fluctuations induced by internal waves in the ocean," J. Acoust. Soc. Am. 60, 795-800 (1976).